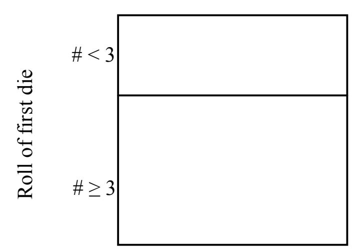
PROBABILITY AND EXPECTATION 10.1.1 – 10.1.2

Students use basic counting principles to compute probabilities, and develop methods for calculating expected value. They compare "expected value" to "most likely outcome" to understand the differences between the two. This leads students to the question of "fairness" in games of chance. Additionally, students review tree diagrams and area models, and solve more challenging probability problems. For further information see the Math Notes boxes following problems 10-9 on page 502 and 10-37 on page 510.

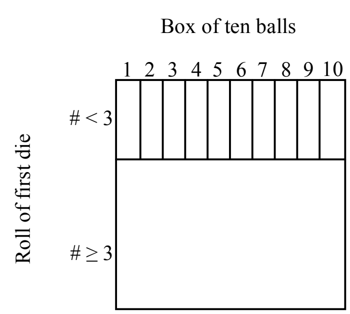
Example 1

Deniz is trying to create a fair game for the Math Club's booth at the spring carnival. A player first rolls a die. If the outcome is less than three, the player chooses a Ping Pong ball from a box containing ten balls, numbered one through ten. If the number on the Ping Pong ball is also less than three, the player wins a prize. If the player's die has a number greater than or equal to three, the player chooses a Ping Pong ball from a different box containing only four balls numbered one through four. If that number is also greater than or equal to three, the player wins a different level of prize. Of course, in order to be sure the game is fair she needs to know the probabilities of each of the outcomes. Draw an area diagram that shows all the outcomes with their probabilities.

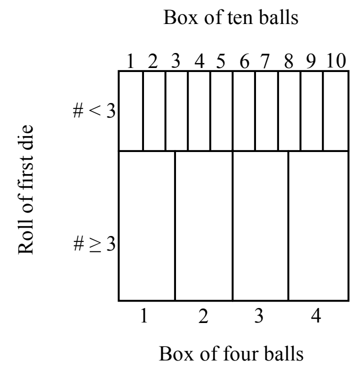


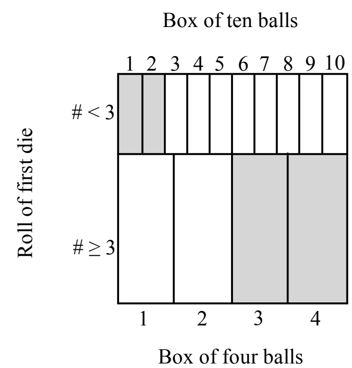
The game starts with the player rolling a die. If the player gets a number less than three (< 3) he will do something different than if he gets a number greater than or equal to three (≥ 3). There are only two numbers less than three –one and two – out of six possible outcomes on a die. Therefore the probability of getting a number less than three is $\frac{2}{6}=\frac{1}{3}$. We show this on our diagram by splitting the top and bottom sections of the rectangle into one- and two-thirds, respectively. Then we label the left side of the area model with the outcome that represents each region. Just by looking at the area of this square, we can see that we are more likely to get a number greater than or equal to three; it takes up most of the area.

Once the die is rolled, different things happen depending on the outcome of the die. If the player got a number less than three (represented by the top third of the area model) then she chooses a Ping Pong ball from a box of ten balls. That means she has ten equally likely possibilities. We will divide the top of the rectangle into ten equal sections and label them across the top of the diagram. Each ball has a probability of $\frac{1}{10}$ of occurring. If we did not get a number less than three (i.e., we are in the bottom two-thirds of the area model), then we choose from a box of only four balls, numbered one through four. Each of these balls is also equally likely, so each will occur with a probability of $\frac{1}{4}$. The first figure below right shows the bottom portion of the model divided into four equally likely sections.



Now the question is, what are the probabilities of each of these outcomes occurring? If we think of this model as a 1 by 1 square, then the left side of the square was split into two pieces, one with a length of one-third, the other a length of two-thirds. Across the top of the one-third portion, the side is divided into ten equal portions, each measuring one-tenth on that side. The area of each skinny rectangle across the top is $\left(\frac{1}{3}\right)\left(\frac{1}{10}\right)=\frac{1}{30}$. The area of each of the four rectangles across the bottom is $\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)=\frac{1}{6}$. Now we can find the probability of winning a prize. The player wins a prize two ways. One way is if the player gets numbers less than three in both instances, with the die and with a Ping Pong ball. The other way to win a prize is if the die and the Ping Pong ball are both greater than or equal to three. The shaded region on the model below right represents the events where the player wins a prize.



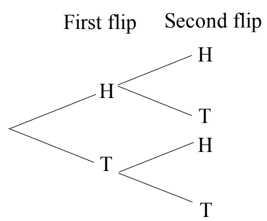


For some students it is helpful, at this point, to think of this as a dartboard problem. Finding the probability of winning a prize is the same as the probability of hitting the shaded portion of this "dartboard." The area of the shaded portion is $\frac{2}{30}+\frac{2}{6}=\frac{2}{30}+\frac{10}{30}=\frac{12}{30}=\frac{2}{5}$. Since the probability of winning is $\frac{2}{5}$, this means the probability of losing is $\frac{3}{5}$. This is not quite a fair game since a player is more likely to lose than to win. A fair game would be one in which a player is just as likely to win as to lose.

Example 2

In a simple coin game, a player flips a coin twice. If both outcomes are heads, the player gets one point. If both outcomes are tails the player gets two points. If the two flips have different outcomes (i.e., heads-tails or tails-heads), the player gets zero points. What is the most likely outcome? What is the expected number of points a player is going to get per play?

To begin, we will calculate the probabilities of each outcome. There are several ways to do this, including an area model as above, or with a tree diagram. Here we will use a tree diagram to model that method. Each branch of the tree illustrates the possible outcomes. Here the first flip of the coin give us two possible outcomes: heads or tails. These outcomes are equally likely, and occur with a probability of one-half. The second flip has the same two equally likely outcomes, each again with a probability of one-half. We read along each branch to see all the possible events: HH, HT, TH, and TT. Each of these events is equally likely with a probability of one-fourth of occurring. In terms of points for this game, the probability of getting one point is one-fourth, the probability of getting two points is also one-fourth, but the probability of getting zero points is one-half. Therefore, the most likely outcome for this game is to get zero points.



The most likely outcome for one flip of the coin, however, is not the same as the expected value. The expected value is what we would expect to get, on average, if we played the game several times. For instance, if we played this game one hundred times, we would expect to get one point about 25 times (one-fourth of the time), two points about 25 times, and zero points about 50 times. This would give a grand total of 75 points for the one hundred games. This means the score, on average, per flip of the coin, is 0.75 or $\frac{3}{4}$ points. This is the expected value: $\frac{3}{4}$ points. Note: It is not possible to get the expected value on any single play of the game.

Problems

When the chef at Honeypenney's House of Pasta prepares side dishes, she does not put much thought into it. She randomly chooses one of three vegetables, either broccoli, carrots or peas. Then she randomly chooses a type of bread stick, either garlic or rosemary.

1. Use a tree diagram to show all the possible pairings of vegetable with a bread stick.

2. Using your tree diagram, what is the probability of getting broccoli with garlic breadsticks?

3. What is the probability of getting carrots and rosemary bread sticks?

4. Suppose you really want garlic breadsticks so you slip the waiter a tip to be sure you get them. If you are certain you will get garlic breadsticks, what is the probability your order will be peas with garlic breadsticks?

A red die and a green die are rolled. What is the probability of:

5. The sum being greater than 7?

6. The sum being a multiple of 3?

7. The sum being less than 12?

A spinner has 40% of it colored blue. Red and green make up the rest of the spinner with red and green taking up the same amount of space.

8. What is the probability of the spinner landing on red?

9. Draw an area diagram representing spinning the spinner twice. What is the probability of getting the same color on both spins?

10. Suppose you know that the spinner landed on the same color twice. What is the probability that the color was red?

11. Suppose you know that at least one time the spinner landed on blue when it was spun twice. What is the probability that the other color was also blue?

12. In a dice game, the player rolls a die. If he gets a one or two, he stops. If he gets any other number, the player gets a point and rolls one more time. If he gets a one or two on the second roll he adds no more points to his score; any other number and he adds one more point and his turn is over. What is the probability of getting:

    a. zero points? b. one point? c. two points?

13. What is the expected value of the game described in the last problem?

The Health Department is developing a two-stage procedure to relieve the symptoms of a rare blood disorder. The first procedure has a 70% probability of giving a person some relief (which is considered a success). If the first procedure does succeed, the second procedure has a 90% probability of success. If the first procedure does not succeed, the second procedure has only a 40% probability of success. What is the probability that:

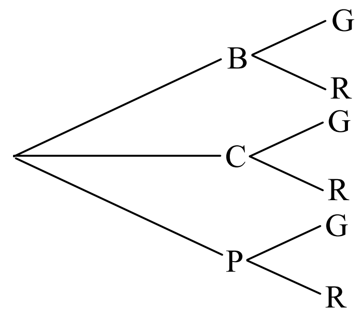
14. Both procedures are successful?

15. Both procedures fail?

16. The first fails and the second succeeds?

17. If you know the second procedure was successful, what is the probability that the first procedure was successful?

Answers



1. Combinations are BG, BR, CG, CR, PG, PR.
2. $\frac{1}{6}$
3. $\frac{1}{6}$
4. $\frac{1}{3}$
5. $\frac{15}{36} = \frac{5}{12}$
6. $\frac{14}{36} = \frac{7}{18}$
7. $\frac{35}{36}$
8. 30% or $\frac{3}{10}$
9. 0.34
10. $\frac{9}{34} \approx 0.26$
11. 0.25
12. a. $\frac{1}{3}$ b. $\frac{2}{9}$ c. $\frac{4}{9}$
13. ≈ 1.11

See figure at right for problems 14 – 17.

14. 0.63
15. 0.18
16. 0.12
17. 0.84

	Blue 40%	Red 30%	Green 30%
Blue 40%	0.16	0.12	0.12
Red 30%	0.12	0.09	0.09
Green 30%	0.12	0.09	0.09

PRINCIPLES OF COUNTING 10.2.1 and 10.2.5

Students take on challenging problems using the Fundamental Principle of Counting, permutations, and combinations to compute probability. These techniques are essential when the sample space is too large to model or to count. For further information see the Math Notes boxes following problems 10-61 on page 516, 10-75 on page 521, 10-88 on page 525, 10-106 on page 530, and 10-144 on page 543.

Example 1

Twenty-three people have entered the pie-eating contest at the county fair. The first place pie-eater (the person eating the most pies in fifteen minutes) wins a pie each week for a year. Second place will receive new baking ware to make his/her own pies, and third place will receive the *Sky High Pies* recipe book. How many different possible top finishers are there?

Since the prizes are different for first, second, and third place, the order of the top finishers matters. We can use a decision chart to determine the number of ways we can have winners. How many different people can come in first? Twenty-three. Once first place is "chosen" (i.e., removed from the list of contenders) how many people are left to take second place? Twenty-two. This leaves twenty-one possible third place finishers. Just as with the branches on the tree diagram, we multiply these numbers to determine the number of arrangements: 23(22)(21) = 10,626. How would you like to draw **that** tree diagram!

23	22	21
First	Second	Third

Example 2

Fifteen students are participating in a photo-shoot for a layout in the new journal *Mathmaticious*. In how many ways can you arrange:

a. Eight of them? b. Two of them? c. Fifteen of them?

We can use a decision chart for each of these situations, but there is another, more efficient method for answering these questions. An arrangement of items where order matters is called a **permutation**, and in this case, since changing the students' order changes the way the picture will look, the order does matter.

With a permutation, you need to know the total number things to be arranged (in this case $n = 15$ students) and how many will be taken (r) at a time. The formula for a permutation is ${}_nP_r = \frac{n!}{(n-r)!}$. In part (a), we have 15 students taken eight at a time. The number of permutations is:

$${}_{15}P_8 = \frac{15!}{(15-8)!} = \frac{15!}{7!} = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 259,459,200$$

In part (b) the solution becomes:

$$_{15}P_2 = \frac{15!}{(15-2)!} = \frac{15!}{13!} = 15 \cdot 14 = 210$$

Part (c) poses a new problem:

$$_{15}P_{15} = \frac{15!}{(15-15)!} = \frac{15!}{0!}$$

What is 0! ? "Factorial" means to calculuate the product of the integers from the given value down to one. How can we compute 0! ? If it equals zero, we have a problem because part (c) would not have an answer (dividing by zero is undefined). But this situations must have an answer. In fact, if we used a decision chart to determine how many ways the 15 people can line up, we would find that there are 15! arrangements. Therefore, if $_{15}P_{15} = 15!$ then 0! = 1. This is another case of mathematicians defining elements of mathematics to fit their needs. 0! is **defined** to equal 1 so that other mathematics makes sense.

Example 3

In the annual homecoming parade, three students get to ride on the lead float. Seven students are being considered for this coveted position. How many ways can three students be chosen for this honor?

All three students who are selected will ride on the lead float, but whether they are the first, second, or third student selected does not matter. In a case where the order of the selections does not matter, such an arrangement is called a **combination**. This means, if the students were labeled A, B, C, D, E, and F, choosing A, B, and then C would be essentially the same as choosing B, C, and then A. In fact, all the arrangements of A, B, and C could be lumped together. This makes the number of combinations much smaller than the number of permutations. The symbol for a combination is $_nC_r$ where n is the total number of items under consideration, and r is the number of items we will choose. It is often read as "*n* choose *r*." In this problem we have $_7C_3$, 7 choose 3. The formula is similar to the formula for a permutation, but we must divide out the similar groups.

$$_nC_r = \frac{n!}{(n-r)!r!}$$

Here we have:

$$_7C_3 = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 35$$

Problems

Simplify the following expressions.

1. $10!$
2. $\frac{10!}{3!}$
3. $\frac{35!}{30!}$
4. $\frac{88!}{87!}$
5. $\frac{72!}{70!}$
6. $\frac{65!}{62!3!}$
7. ${}_8P_2$
8. ${}_{15}P_0$
9. ${}_9P_9$
10. ${}_{12}C_4$
11. ${}_5C_0$
12. ${}_{32}C_{32}$

Solve the following problems.

13. How many ways can you arrange the letters from the word “KAREN”?
14. How many ways can you arrange the letters from the word “KAREN” if you want the arrangement to begin with a vowel?
15. All standard license plates in Alaska start with three letters followed by three digits. If repetition is allowed, how many different license plates are there?
16. For $3.99, The Creamery Ice Cream Parlor will put any three different flavored scoops, out of their 25 flavors of ice cream, into a bowl. How many different “bowls” are there? (Note: A bowl of chocolate, strawberry, and vanilla is the same bowl as a bowl of chocolate, vanilla and strawberry.)
17. Suppose those same three scoops of ice cream are on a cone. Now how many arrangements are there? (Note: Ice cream on a cone must be eaten “top down” because you cannot eat the bottom or middle scoop out, and keep the cone intact.)
18. A normal deck of playing cards contains 52 cards. How many five-card poker hands can be made?
19. How many ways are there to make a full house (three of one kind, two of another)?
20. What is the probability of getting a full house (three of one kind of card and two of another)? Assume a standard deck and no wild cards.

For problems 21–25, a bag contains 36 marbles. There are twelve blue marbles, eight red marbles, seven green marbles, five yellow marbles and four white marbles. Without looking, you reach into the bag and pull out eight marbles. What is the probability you pull out:

21. All blue marbles?
22. Four blue and four white marbles?
23. Seven green and one yellow marble?
24. At least one red and at least two yellow?
25. No blue marbles?

Answers

1.	3,628,800	7.	56	13.	$5! = 120$
2.	604,800	8.	1	14.	$2(4!) = 48$
3.	38,955,840	9.	362,880	15.	$(26)(26)(26)(10)(10)(10) = 17{,}576{,}000$
4.	88	10.	495	16.	${}_{25}C_3 = 2300$
5.	5,112	11.	1	17.	${}_{25}P_3 = 13{,}800$ (On a cone, order matters!)
6.	43,680	12.	1	18.	${}_{52}C_5 = 2{,}598{,}960$

19. This is tricky and tough! There are 13 different "types" of cards: twos, threes, fours, ..., Jacks, Queens, Kings, and Aces. We need to choose which of the 13 we want three of (${}_{13}C_1$). Once we choose what type (for example, we pick Jacks) then we need to choose which three out of the four to take (${}_{4}C_3$). Then from the remaining 12 types, we choose which type to have two of (${}_{12}C_1$). Then again we need to choose which two out of the four (${}_{4}C_2$). This gives us $\left({}_{13}C_1\right)\cdot\left({}_{4}C_3\right)\cdot\left({}_{12}C_1\right)\cdot\left({}_{4}C_2\right) = 3{,}744$.

20. We have already calculated the numbers we need in problems 18 and 19 so: $\frac{3{,}744}{2{,}598{,}960} \approx 0.0014$.

21. Each time we reach in and pull out 8 marbles, order does not matter. The number of ways to do this is ${}_{36}C_8$. This is the number in the sample space, i.e., the denominator. How many ways can we pull out all blue? ${}_{12}C_8$. Therefore the probability is $\frac{{}_{12}C_8}{{}_{36}C_8} \approx 0.0000164$.

22. Same denominator. Now we want to choose 4 from the 12 blue, ${}_{12}C_4$, and 4 from the 4 whites, ${}_{4}C_4$. $\frac{{}_{12}C_4 \cdot {}_{4}C_4}{{}_{36}C_8} \approx 0.0000164$, the same answer!

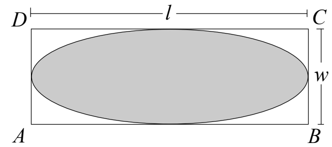
23. Seven green: ${}_{7}C_7$, one yellow: ${}_{5}C_1$. $\frac{{}_{7}C_7 \cdot {}_{5}C_1}{{}_{36}C_8} \approx 0.0000001652$

24. Here we have to get at least one red: ${}_{8}C_1$, and at least two yellow: ${}_{5}C_2$, but the other five marbles can come from the rest of the pot: ${}_{33}C_5$. Therefore, $\frac{{}_{8}C_1 \cdot {}_{5}C_2 \cdot {}_{33}C_5}{{}_{36}C_8} \approx 0.627$.

25. To get no blue marbles means we want all eight from the other 24 non-blue marbles. $\frac{{}_{24}C_8}{{}_{36}C_8} \approx 0.0243$.

SAT PREP

1. In the rectangle *ABCD* at right, the area of the shaded region is given by $\frac{\pi l w}{6}$. If the area of the shaded region is 7π, what is the total area, to the nearest whole number, of the unshaded regions of the rectangle *ABCD*?



a. 14 b. 15 c. 20 d. 22 e. 25

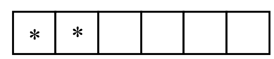
2. Consider the following equations:

$$a = p^3 - 0.61$$
$$b = p^2 - 0.61$$
$$c = (p - 0.61)^2$$

If p is a negative integer, what is the ordering of a, b, and c from least to greatest?

a. $c < a < b$ b. $a < c < b$ c. $b < a < c$ d. $a < b < c$ e. $c < b < a$

3. The figure at right represents six offices that will be assigned randomly to six different employees, one employee per office. If Maryanne and Ginger are two of the six employees, what is the probability that they will be assigned an office indicated with an *?



a. $\frac{1}{6}$ b. $\frac{1}{8}$ c. $\frac{1}{15}$ d. $\frac{2}{15}$ e. $\frac{1}{30}$

4. Raul needed wire pieces 7 inches long. He cut as many as he possibly could from a wire 6 feet long. What is the total length of the wire that is left over?

a. 2 inches b. 3 inches c. 4 inches d. 5 inches e. 8 inches

5. The n^{th} term of a sequence is defined to be $5n + 2$. The 35^{th} term is how much greater than the 30^{th} term?

a. 5 b. 18 c. 25 d. 36 e. 40

6. Matilda remembers only the first four digits of a seven-digit phone number. She is certain that none of the last three digits is zero. If she dials the first four digits, then dials the last three digits randomly from the non-zero digits, what is the probability that she will dial the correct number?

7. Let $a \vartriangle b$ be defined as $\frac{1}{a} + b$ where $a \neq 0$. What is the value of $6 \vartriangle 7$?

8. If $4xy + 1 = 1$, what is the value of xy?

9. Eight consecutive integers are arranged in order from smallest to the largest. If the sum of the first four integers is 206, what is the sum of the last four integers?

10. If the points $A(4, 1)$, $B(4, 8)$, and $C(-3, 8)$ form the vertices of a triangle, what is the area of the triangle?

Answers

1. C
2. D
3. C
4. A
5. C
6. $\frac{1}{729}$
7. $\frac{43}{6}$
8. 0
9. 222
10. 24.5